

SIMPLE TRANSITIVE 2-REPRESENTATIONS FOR SOME 2-SUBCATEGORIES OF SOERGEL BIMODULES

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ABSTRACT. We classify simple transitive 2-representations of certain 2-subcategories of the 2-category of Soergel bimodules over the coinvariant algebra in Coxeter types B_2 and $I_2(5)$. In the $I_2(5)$ case it turns out that simple transitive 2-representations are exhausted by cell 2-representations. In the B_2 case we show that, apart from cell 2-representations, there is a unique, up to equivalence, additional simple transitive 2-representation and we give an explicit construction of this 2-representation.

1. INTRODUCTION AND DESCRIPTION OF THE RESULTS

Classification problems are, historically, the main driving force of representation theory. The desire to understand and, in particular, classify certain classes of representations of a given group or algebra was behind the majority of research in the general area of representation theory since its birth.

The abstract 2-representation theory, which originated in [BFK, CR, KL, Ro], studies functorial actions of 2-categories. The “finite-dimensional” part of this theory, that is 2-representation theory of finitary 2-categories, was systematically developed in the series [MM1, MM2, MM3, MM4, MM5, MM6] and continued in [Xa, Zh1, Zh2]. In particular, the paper [MM5] proposes a very good candidate for the notion of a “simple” 2-representation, called a *simple transitive* 2-representation. In the same paper one finds a classification of such 2-representations for a special class of finitary 2-categories with involution which includes the 2-category of Soergel bimodules over the coinvariant algebra of the symmetric group. This is extended in [MM6] to more general 2-categories and (slightly) more general classes of 2-representations. Some nice applications of these classification results were obtained in [KM1].

The classification of simple transitive 2-representations for some “smallest” 2-categories which do not fit the setup and methods of [MM5, MM6] was completed in [MZ, Zi]. All the results mentioned above have, however, one common feature. It turns out that in all these cases the simple transitive 2-representations are exhausted by the so-called *cell 2-representations* defined and studied in [MM1]. So far there was only one, quite artificial, example of a family of simple transitive 2-representations which are not equivalent to cell 2-representations, constructed in [MM5, Subsection 3.2] using transitive group actions.

In the present paper we study simple transitive 2-representations of a certain 2-subquotient \mathcal{Q}_n of the 2-category of Soergel bimodules for the dihedral group $D_{2,n}$, where $n \geq 3$. For $n = 3$, this 2-category fits into the setup of [MM5] and hence the classification result from [MM5] directly applies. For $n > 3$, the 2-category \mathcal{Q}_n must be studied by other methods.

We show that every simple transitive 2-representation of \mathcal{Q}_5 is equivalent to a cell 2-representation, see Theorem 5. We also show that, apart from cell 2-representations, there is a unique, up to equivalence, simple transitive 2-representation of \mathcal{Q}_4 , see Theorem 12. The corresponding 2-representation is explicitly constructed in Subsection 5.8. This subsection is the heart of this paper. Construction of this new 2-representation is based on the careful interplay of several category-theoretic tricks. The case $n > 5$ seems, at the moment, computationally too difficult.

The paper is organized as follows: in Section 2 we collect all necessary preliminaries from 2-representation theory. In Section 3 we recall the definition and combinatorics of the 2-category of Soergel bimodules over a dihedral group and define the 2-category \mathcal{Q}_n , our main object of study. Theorem 5 is proved in Section 4. Theorem 12 is proved in Section 5.

Acknowledgment. A major part of the research which led to this paper was done during the visit of the first author to Uppsala University which was partially supported by the Swedish Research Council. Hospitality of Uppsala University and financial support of the Swedish Research Council are gratefully acknowledged. We thank Aaron Chan for comments on the original version. We are grateful to the referee for numerous comments and suggestions which helped to improve exposition.

2. GENERALITIES ON 2-CATEGORIES AND 2-REPRESENTATIONS

2.1. Notation and conventions. We work over \mathbb{C} and write \otimes for $\otimes_{\mathbb{C}}$. By a module we mean a *left* module. Maps are composed from right to left.

2.2. Finitary and fiat 2-categories. We refer the reader to [Le, Mac, Maz] for generalities on 2-categories.

A 2-category is a category enriched over the monoidal category **Cat** of small categories. Thus, a 2-category \mathcal{C} consists of objects (denoted by Roman lower case letters in a typewriter font), 1-morphisms (denoted by capital Roman letters), and 2-morphisms (denoted by Greek lower case letters), composition of 1-morphisms, horizontal and vertical compositions of 2-morphisms (denoted \circ_0 and \circ_1 respectively), identity 1-morphisms and identity 2-morphisms. These must satisfy the obvious collection of axioms. For a 1-morphism F , we denote by id_F the corresponding identity 2-morphism. As usual, we often write $F(\alpha)$ for $\text{id}_F \circ_0 \alpha$ and α_F for $\alpha \circ_1 \text{id}_F$.

A 2-category \mathcal{C} is called *finitary* if, for each pair (i, j) of objects in \mathcal{C} , the category $\mathcal{C}(i, j)$ is an idempotent split, additive and Krull-Schmidt \mathbb{C} -linear category with finitely many isomorphism classes of indecomposable objects and finite dimensional morphism spaces; moreover, all compositions must be compatible with these additional structures, see [MM1] for details.

A finitary 2-category \mathcal{C} is called *fiat* if it has a weak involution \star together with adjunction 2-morphisms satisfying the usual axioms of adjoint functors, for each pair (F, F^\star) of 1-morphisms, see [MM1] for details.

2.3. 2-representations. For a finitary 2-category \mathcal{C} , we consider the 2-category $\mathcal{C}\text{-afmod}$ of all *finitary 2-representations* of \mathcal{C} as defined in [MM3]. Objects of $\mathcal{C}\text{-afmod}$ are strict functorial actions on idempotent split, additive and Krull-Schmidt \mathbb{C} -linear categories with finitely many isomorphism classes of indecomposable objects

and finite dimensional morphism spaces. Furthermore, 1-morphisms in $\mathcal{C}\text{-afmod}$ are strong 2-natural transformations and 2-morphisms are modifications.

Similarly we can consider the 2-category $\mathcal{C}\text{-mod}$ of all *abelian 2-representations* of \mathcal{C} , that is functorial actions on categories equivalent to module categories over finite dimensional algebras, see [MM3] for details. We have the diagrammatically defined abelianization 2-functor

$$\overline{\cdot} : \mathcal{C}_A\text{-afmod} \rightarrow \mathcal{C}_A\text{-mod},$$

see [MM2, Subsection 4.2] for more details.

Two 2-representations are said to be *equivalent* if there is a strong 2-natural transformation Φ between them such that the restriction of Φ to each object in \mathcal{C} is an equivalence of categories.

A finitary 2-representation \mathbf{M} of \mathcal{C} will be called *transitive* provided that, for each indecomposable objects X and Y in $\coprod_i \mathbf{M}(\mathbf{i})$, there is a 1-morphism F in \mathcal{C} such that Y is isomorphic to a direct summand of the object $\mathbf{M}(F)X$. A transitive 2-representation \mathbf{M} is called *simple transitive* if $\coprod_i \mathbf{M}(\mathbf{i})$ does not have any non-zero proper \mathcal{C} -invariant ideals.

Similarly one can define the notion of *transitive (based) module* over any positively based algebra A in the sense of [KM2, Section 9]. Here transitivity means that the basis of V has the property that, for any elements v and w in this basis, there is an element a in the basis of A such that v appears with a non-zero coefficient in aw .

For simplicity, we often use the module notation $F X$ instead of the corresponding representation notation $\mathbf{M}(F) X$.

2.4. Combinatorics. For a finitary 2-category \mathcal{C} , we denote by $\mathcal{S}[\mathcal{C}]$ the corresponding *multisemigroup* as defined in [MM2, Section 3]. By \leq_L , \leq_R and \leq_J we denote the corresponding *left*, *right* and *two-sided* orders on $\mathcal{S}[\mathcal{C}]$. Equivalence classes for these orders are called *cells*. For simplicity, we will abuse the language and say “cells of \mathcal{C} ” instead of “cells of $\mathcal{S}[\mathcal{C}]$ ”.

If \mathcal{J} is a two-sided cell in $\mathcal{S}[\mathcal{C}]$, then the 2-category \mathcal{C} is called *\mathcal{J} -simple* provided that any non-zero two-sided 2-ideal of \mathcal{C} contains the identity 2-morphisms for all 1-morphisms in \mathcal{J} , see [MM2].

2.5. Cell 2-representations. For $\mathbf{i} \in \mathcal{C}$, we denote by $\mathbf{P}_{\mathbf{i}} := \mathcal{C}_A(\mathbf{i}, -)$ the corresponding *principal* 2-representation. For a left cell \mathcal{L} in $\mathcal{S}[\mathcal{C}]$, we denote by $\mathbf{C}_{\mathcal{L}}$ the corresponding *cell* 2-representation, see [MM1, MM2] for details.

2.6. Matrices in the Grothendieck group. For a finitary 2-category and a finitary 2-representation \mathbf{M} of \mathcal{C} , let us fix a complete and irredundant list of representatives of isomorphism classes of indecomposable objects in $\coprod_i \mathbf{M}(\mathbf{i})$. Then, for any 1-morphism F , we have the corresponding matrix $\llbracket F \rrbracket$ which counts multiplicities in direct sum decompositions of the images of indecomposable objects under F .

If \mathcal{C} is fiat, then each $\overline{\mathbf{M}}(F)$ is exact and we have the matrix $\llbracket F \rrbracket$ which counts composition multiplicities of the images of simple objects under F . By adjunction, the matrix $\llbracket F \rrbracket$ is transposed to the matrix $\llbracket F^* \rrbracket$.

3. THE 2-CATEGORY \mathcal{Q}_n

3.1. Soergel bimodules for dihedral groups. We refer the reader to [So1, So2, E, EW] for more information and details on Soergel bimodules.

For $n \geq 3$, consider the dihedral group $D_{2,n}$ of symmetries of a regular n -gon in \mathbb{R}^2 with its corresponding *defining* module \mathbb{C}^2 . The group $D_{2,n}$ has Coxeter presentation

$$D_{2,n} = \langle s, t : s^2 = t^2 = (st)^n = e \rangle.$$

We may assume that, in the defining representation, the elements s and t act via the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos(\frac{2\pi}{n}) & \sin(\frac{2\pi}{n}) \\ \sin(\frac{2\pi}{n}) & -\cos(\frac{2\pi}{n}) \end{pmatrix},$$

respectively. Let \leq denote the Bruhat order on $D_{2,n}$. For $w \in D_{2,n}$, set

$$\underline{w} := \sum_{v \leq w} v.$$

Then $\{\underline{w} : w \in D_{2,n}\}$ is the *Kazhdan-Lusztig* basis of $\mathbb{Z}D_{2,n}$. We denote by w_0 the longest element in $D_{2,n}$.

Let \mathbb{C} be the coinvariant algebra associated to the defining $D_{2,n}$ -module \mathbb{C}^2 . Let, further, \mathbb{C}^s denote the subalgebra of s -invariants in \mathbb{C} and \mathbb{C}^t denote the subalgebra of t -invariants in \mathbb{C} . A *Soergel \mathbb{C} - \mathbb{C} -bimodule* is a \mathbb{C} - \mathbb{C} -bimodule isomorphic to a bimodule from the additive closure of the monoidal category of \mathbb{C} - \mathbb{C} -bimodules generated by

$$\mathbb{C} \otimes_{\mathbb{C}^s} \mathbb{C} \quad \text{and} \quad \mathbb{C} \otimes_{\mathbb{C}^t} \mathbb{C}.$$

Isomorphism classes of indecomposable Soergel bimodules are naturally indexed by $w \in D_{2,n}$ and we denote by B_w a fixed representative from such a class.

Consider a small category \mathcal{C} equivalent to $\mathbb{C}\text{-mod}$. Define the 2-category \mathcal{S}_n of *Soergel bimodules* (associated to \mathcal{C}) as follows:

- \mathcal{S}_n has one object \mathbf{i} , which we can identify with \mathcal{C} ;
- 1-morphisms in \mathcal{S}_n are all endofunctors of \mathcal{C} which are isomorphic to endofunctors given by tensoring with Soergel \mathbb{C} - \mathbb{C} -bimodules;
- 2-morphisms in \mathcal{S}_n are natural transformations of functors (these correspond to homomorphisms of Soergel \mathbb{C} - \mathbb{C} -bimodules).

The 2-category \mathcal{S}_n is fiat.

For $w \in D_{2,n}$, let θ_w denote a fixed representative in the isomorphism class of indecomposable 1-morphisms given by tensoring with B_w . The 2-category \mathcal{S}_n has three two-sided cells

$$\mathcal{J}_e := \{\theta_e\}, \quad \mathcal{J}_{w_0} := \{\theta_{w_0}\}, \quad \mathcal{J}_s := \{\theta_w : w \neq e, w_0\},$$

which are linearly ordered $\mathcal{J}_e \leq_J \mathcal{J}_s \leq_J \mathcal{J}_{w_0}$. We also have four left cells

$$\begin{aligned} \mathcal{L}_e &:= \{\theta_e\}, & \mathcal{L}_{w_0} &:= \{\theta_{w_0}\}, & \mathcal{L}_s &:= \{\theta_w : w \neq w_0, ws < w\}, \\ \mathcal{L}_t &:= \{\theta_w : w \neq w_0, wt < w\}. \end{aligned}$$

The left order on these left cells is given by

$$\mathcal{L}_e \leq_L \mathcal{L}_s \leq_L \mathcal{L}_{w_0} \quad \text{and} \quad \mathcal{L}_e \leq_L \mathcal{L}_t \leq_L \mathcal{L}_{w_0},$$

with \mathcal{L}_s and \mathcal{L}_t being incomparable. Applying $w \mapsto w^{-1}$, one obtains right cells

$$\begin{aligned} \mathcal{R}_e &:= \{\theta_e\}, & \mathcal{R}_{w_0} &:= \{\theta_{w_0}\}, & \mathcal{R}_s &:= \{\theta_w : w \neq w_0, sw < w\}, \\ & & \mathcal{R}_t &:= \{\theta_w : w \neq w_0, tw < w\}. \end{aligned}$$

and the right order

$$\mathcal{R}_e \leq_R \mathcal{R}_s \leq_R \mathcal{R}_{w_0} \quad \text{and} \quad \mathcal{R}_e \leq_R \mathcal{R}_t \leq_R \mathcal{R}_{w_0},$$

with \mathcal{R}_s and \mathcal{R}_t being incomparable.

For the decategorification $[\mathcal{S}_n]$, we have an isomorphism

$$[\mathcal{S}_n](\mathbf{i}, \mathbf{i}) \cong \mathbb{Z}D_{2 \cdot n}$$

which sends $[\theta_w]$ to \underline{w} , see [So2] (and also [E]).

3.2. The 2-subquotient \mathcal{Q}_n of \mathcal{S}_n . Let \mathcal{I} be the 2-ideal of \mathcal{S}_n generated by $\text{id}_{\theta_{w_0}}$. Denote by \mathcal{Q}'_n the 2-subcategory of $\mathcal{S}_n/\mathcal{I}$ defined as follows:

- \mathcal{Q}'_n has the same objects as $\mathcal{S}_n/\mathcal{I}$;
- 1-morphisms in \mathcal{Q}'_n are all 1-morphisms in $\mathcal{S}_n/\mathcal{I}$ which are isomorphic to direct sums of the 1-morphisms θ_e and $\theta_w \in \mathcal{L}_s \cap \mathcal{R}_s$;
- 2-morphisms in \mathcal{Q}'_n are inherited from $\mathcal{S}_n/\mathcal{I}$.

The 2-category \mathcal{Q}'_n inherits from \mathcal{S}_n the structure of a fiat 2-category. Abusing notation, we will denote indecomposable 1-morphisms in \mathcal{Q}'_n by the same symbols as for \mathcal{S}_n . Directly from the construction it follows that \mathcal{Q}'_n has two two-sided cells which are, at the same time, both left and right cells:

$$\mathcal{J}_e = \mathcal{L}_e = \mathcal{R}_e = \{\theta_e\}, \quad \mathcal{J}_s = \mathcal{L}_s = \mathcal{R}_s = \{\theta_w : w \neq w_0; ws < w, sw < w\}$$

such that \mathcal{J}_e is the minimum element with respect to the left, right and two-sided orders and $\mathcal{J}_s := \mathcal{J}_s$ is the maximum element with respect to the left, right and two-sided orders.

We define the 2-category \mathcal{Q}_n as the quotient of \mathcal{Q}'_n by the unique 2-ideal which is maximal in the set of all 2-ideals which do not contain any id_F , for non-zero F . This ideal exists, see e.g. [MM5, Lemma 4], however, we do not know any explicit set of generators for it. The 2-category \mathcal{Q}_n is the main object of study in the present paper. By construction, the 2-category \mathcal{Q}_n is fiat and \mathcal{J} -simple.

We also denote by \mathcal{A}_n the 2-full 2-subcategory of \mathcal{Q}_n with 1-morphisms in the additive closure of θ_e and θ_s .

3.3. Simple transitive 2-representations of \mathcal{Q}_3 . For $n = 3$, the two-sided cell \mathcal{J}_s contains only one element. Therefore $\mathcal{Q}_3 = \mathcal{A}_3$ fits into the general setup of [MM5]. Consequently, by [MM5, Theorem 18], each simple transitive 2-representation of \mathcal{Q}_3 is equivalent to a cell 2-representation. In fact, from [MM3, Theorem 13] it follows that the 2-category \mathcal{Q}_3 is biequivalent to the 2-category of Soergel bimodules for the symmetric group S_2 .

For the cell \mathcal{L}_e , the corresponding cell 2-representation is, roughly speaking, given by the obvious functorial action on $\mathbb{C}\text{-mod}$, where θ_s acts as the zero functor.

For the cell \mathcal{L}_s , the corresponding cell 2-representation is, roughly speaking, given by a functorial action of \mathcal{Q}_3 on the category of projective modules over the algebra $D = \mathbb{C}[x]/(x^2)$ of dual numbers. Here θ_s acts via tensoring with the projective D - D -bimodules $D \otimes D$. This is a special case of a very general picture which we describe in the next subsection.

3.4. Some 2-categories similar to \mathcal{Q}_3 . Let A be a finite dimensional algebra and B a subalgebra of A . Assume the following:

- (I) A is local, commutative and Frobenius;
- (II) B is Frobenius;
- (III) A is free of rank two, as a B -module.

Let \mathcal{A} be a small category equivalent to $A\text{-mod}$. Let \mathcal{G} denote the 2-category defined as follows:

- \mathcal{G} has one object \mathbf{i} , which we identify with $A\text{-mod}$;
- 1-morphisms in \mathcal{G} are endofunctors in $A\text{-mod}$ in the additive closure of the identity functor and the functor of tensoring with $A \otimes_B A$;
- 2-morphisms in \mathcal{G} are natural transformations of functors.

This is well-defined because of the isomorphism

$$(1) \quad A \otimes_B A \otimes_A A \otimes_B A \cong (A \otimes_B A) \oplus (A \otimes_B A)$$

which follows from (III).

We denote by F a 1-morphism in \mathcal{G} given by tensoring with $A \otimes_B A$. Then we have $F^2 \cong F \oplus F$ from (1).

Lemma 1. *The 2-category \mathcal{G} is fiat.*

Proof. The computation in the proof of [MM6, Proposition 24] shows that there is an isomorphism of A - A -bimodules as follows:

$$\text{Hom}_{\mathbb{C}}(A \otimes_B A, \mathbb{C}) \cong A \otimes_B A.$$

Therefore we can define the weak involution \star using $\text{Hom}_{\mathbb{C}}(-, \mathbb{C})$. We have $F^{\star} \cong F$.

To prove existence of adjunction morphisms, it is enough to show that F is self-adjoint. Note that F is the composition of the restriction from A to B followed by induction from B to A . As both are symmetric, we compute the right adjoint of the restriction:

$$\begin{aligned} \text{Hom}_B({}_B A_A, B) &\cong \text{Hom}_B({}_B A_A, \text{Hom}_{\mathbb{C}}(B, \mathbb{C})) \\ &\cong \text{Hom}_{\mathbb{C}}(B \otimes_B {}_B A_A, \mathbb{C}) \\ &\cong \text{Hom}_{\mathbb{C}}({}_B A_A, \mathbb{C}) \\ &\cong {}_A A_B, \end{aligned}$$

where we used $\text{Hom}_{\mathbb{C}}(A, \mathbb{C}) \cong A$ as A is symmetric and $\text{Hom}_{\mathbb{C}}(B, \mathbb{C}) \cong B$ as B is symmetric. This shows that restriction is biadjoint to induction. Therefore F is self-adjoint. \square

Recall also the 2-category \mathcal{C}_D from [MM1, Subsection 7.3], defined as follows: let \mathcal{D} be a small category equivalent to $D\text{-mod}$.

- \mathcal{C}_D has one object \mathbf{i} , which we identify with ;

- 1-morphisms in \mathcal{C}_D are endofunctors of \mathcal{D} in the additive closure of the identity functor and the functor of tensoring with $D \otimes_{\mathbb{C}} D$;
- 2-morphisms are natural transformations of functors.

Proposition 2. *The 2-category \mathcal{G} has a unique ideal \mathcal{J} which is maximal in the set of all ideals that do not contain the identity 2-morphisms of non-zero one-morphisms. The quotient \mathcal{G}/\mathcal{J} is biequivalent to \mathcal{C}_D .*

Our description of the cell 2-representation $\mathbf{C}_{\mathcal{L}_s}$ for the 2-category \mathcal{Q}_3 follows directly from Proposition 2.

Lemma 3. *Let $L \in \mathcal{A}$ be a simple, then $F(L)$ is indecomposable of length two.*

Proof. As both A and B are local and commutative, see (I), they both are basic and have a unique simple module. Therefore, the simple A -module L is 1-dimensional and restricts to a simple B -module, which we call L' . By adjunction

$$\mathrm{Hom}_A(A \otimes_B L', L) \cong \mathrm{Hom}_B(L', L')$$

and the latter space is 1-dimensional as L' is 1-dimensional. Therefore the module $A \otimes_B L' \cong F(L)$ has simple top. The module $F(L)$ has length two as A is free over B of rank two. The claim follows. \square

Note that $\mathrm{End}_A(F(L)) \cong D$, where $x \in D$ corresponds to the nilpotent endomorphism of $F(L)$ which maps the top of $F(L)$ to the socle of $F(L)$. This brings D into the picture. Now we can prove Proposition 2.

Proof of Proposition 2. Consider the additive closure \mathcal{G} of $M = F(L)$. From (1) it follows that the additive category \mathcal{G} is invariant under the action of \mathcal{G} . Let \mathbf{M} denote the restriction 2-representation of \mathcal{G} corresponding to the action of \mathcal{G} on \mathcal{G} . Note that \mathcal{G} is equivalent to the category $D\text{-proj}$ of projective D -modules. The abelianization of \mathcal{G} is equivalent to the additive subcategory generated by L and M , via the functor of taking a projective presentation (we note that the additive subcategory generated by L and M is, in fact, abelian).

Now, F is a self-adjoint functor which maps a simple object L to a projective object M (in the abelianization of \mathcal{G}). By [MM5, Lemma 13], F is given by tensoring with $D \otimes_{\mathbb{C}} D$, via an equivalence of \mathcal{G} with $D\text{-proj}$. Therefore the representation map \mathbf{M} defines a 2-functor Φ from \mathcal{G} to a 2-category biequivalent to \mathcal{C}_D , via an equivalence of \mathcal{G} with $D\text{-proj}$. Moreover, from the classification of simple fiat 2-categories in [MM3, Theorem 13], we have that the image of Φ is, in fact, biequivalent to \mathcal{C}_D .

The kernel of Φ is contained in \mathcal{J} , by construction. Since \mathcal{C}_D is \mathcal{J} -simple, where $\mathcal{J} = \{D \otimes_{\mathbb{C}} D\}$, see [MM1, MM3], we have that the kernel of Φ equals \mathcal{J} and the claim of the proposition follows. \square

Corollary 4. *Let $\mathbf{M} \in \mathcal{G}\text{-afmod}$ and $L \in \overline{\mathbf{M}}(\mathbf{i})$ be simple. Then $F(L)$ is either zero or indecomposable. In case $F(L)$ is indecomposable, then both the top and the socle of $F(L)$ are isomorphic to L , moreover, $F(\mathrm{Rad}(L)/\mathrm{Soc}(L)) = 0$.*

Proof. As \mathcal{G} is fiat with strongly regular two-sided cells, every simple transitive 2-representation of \mathcal{G} is a cell 2-representation by [MM5, Theorem 15]. Hence there are exactly two different simple transitive 2-representation of \mathcal{G} and the matrix (F) for these two 2-representation is (0) or (2).

From [ChMa, Theorem 25] it follows that, for any \mathbf{M} , the matrix (F) has the form

$$\begin{pmatrix} 2E & * \\ 0 & 0 \end{pmatrix},$$

where E is the identity matrix, up to permutation of basis elements. By adjunction, this implies $F(L) = 0$ or $[F(L) : L] = 2$. In the latter case exactness of F yields that F annihilates all simple subquotients of $F(L)$ which are not isomorphic to L . Therefore, by adjunction, only L can appear in the top or socle of $F(L)$. Now, from indecomposability we get that both socle and top must be isomorphic to L . The claim follows. \square

4. SIMPLE TRANSITIVE 2-REPRESENTATIONS OF \mathcal{Q}_5

4.1. The result. Our main result for the 2-category \mathcal{Q}_5 is the following:

Theorem 5. *Every simple transitive 2-representation of \mathcal{Q}_5 is equivalent to a cell 2-representation.*

4.2. The algebra $[\mathcal{Q}_5](\mathbf{i}, \mathbf{i})$. We identify the ring $R = [\mathcal{Q}_5](\mathbf{i}, \mathbf{i})$ with the corresponding subquotient of $\mathbb{Z}D_{2,5}$.

With this identification, the ring R has basis $\{\underline{e}, \underline{s}, \underline{sts}\}$ and the following multiplication table of $x \cdot y$:

$$(2) \quad \begin{array}{c|c|c|c} x \setminus y & \underline{e} & \underline{s} & \underline{sts} \\ \hline \underline{e} & \underline{e} & \underline{s} & \underline{sts} \\ \hline \underline{s} & \underline{s} & 2\underline{s} & 2\underline{sts} \\ \hline \underline{sts} & \underline{sts} & 2\underline{sts} & 2\underline{s} + 2\underline{sts} \end{array}$$

The \mathbb{C} -algebra $A := \mathbb{C} \otimes_{\mathbb{Z}} R$ is commutative and split semisimple.

The three 1-dimensional representations of A are given by the following table which describes the action of the basis elements in any fixed basis:

$$(3) \quad \begin{array}{c|c|c|c} & V_1 & V_2 & V_3 \\ \hline \underline{e} & 1 & 1 & 1 \\ \hline \underline{s} & 0 & 2 & 2 \\ \hline \underline{sts} & 0 & 1 - \sqrt{5} & 1 + \sqrt{5} \end{array}$$

Cells in A have exactly the same combinatorics as for \mathcal{Q}_5 . Recall, from [KM2, Subsection 5.1], that a subquotient of a transitive based A -module is called *special* provided that it contains a unique maximal (in the absolute value) eigenvalue for the element $\underline{e} + \underline{s} + \underline{sts}$. Special simple A -modules are in bijection with two-sided cells, see [KM2, Subsection 9.4]. For the cell \mathcal{L}_e , the special A -module is V_1 ; for the cell \mathcal{L}_s , the special A -module is V_3 .

4.3. Reduction to the rank two case. Let $\hat{\mathcal{Q}}_5$ be the quotient of \mathcal{Q}_5 by the 2-ideal generated by 1-morphisms in \mathcal{J} . Then the only surviving indecomposable 1-morphism in $\hat{\mathcal{Q}}_5$ is the identity 1-morphism, up to isomorphism. Therefore each simple transitive 2-representation of $\hat{\mathcal{Q}}_5$ is a cell 2-representation by [MM5, Theorem 18].

Let \mathbf{M} be a simple transitive 2-representation of \mathcal{Q}_5 . Recall that \mathcal{Q}_5 is \mathcal{J} -simple. Therefore, if \mathbf{M} is not faithful, it factors through $\hat{\mathcal{Q}}_5$. Consequently, from the

previous paragraph we obtain that \mathbf{M} is a cell 2-representation. Therefore, from now on in this section we may assume that \mathbf{M} is faithful.

Lemma 6. *Each faithful simple transitive 2-representation \mathbf{M} of \mathcal{Q}_5 has rank two and decategorifies to $V_2 \oplus V_3$.*

Proof. Consider the transitive module $V = \mathbb{C} \otimes_{\mathbb{Z}} [\mathbf{M}(\mathbf{i})]$ over the positively based algebra A . As \mathbf{M} is faithful, either V_2 or V_3 must appear as a subquotient of V . Being special, V_3 appears with multiplicity one by [KM2, Section 9]. Since V_2 is not special, it follows that

$$V \cong V_3 \oplus V_2^{\oplus x} \oplus V_1^{\oplus y}.$$

As \underline{sts} must have integral trace, from (3) it follows that $x = 1$. Further, transitivity of \mathbf{M} implies that the matrix M of the action of $\underline{s} + \underline{sts}$ on V (in the basis of indecomposable projectives) is a positive integral matrix. Let N be the matrix of the action of \underline{s} (in the same basis).

The 2-category \mathcal{A}_5 fits into the general framework of [MM5]. Hence any simple transitive 2-representation of \mathcal{A}_5 is a cell 2-representation by [MM5, Theorem 18]. This means that \mathcal{A}_5 has two simple transitive 2-representations and the matrices (θ_s) in these 2-representations are (2) and (0), cf. [Zi, Subsection 6.3] and, also, Subsection 3.4.

If the matrix N is not upper-triangular after some permutation of the basis, then there is a simple transitive 2-representation of \mathcal{A}_5 which is at least of rank 2, a contradiction. Given that N is upper-triangular, the diagonals have to be 2 or 0 by the previous paragraph. As the 1-morphism θ_s is self-adjoint and $\theta_s^2 \cong \theta_s \oplus \theta_s$, for each simple object L in $\overline{\mathbf{M}}(\mathbf{i})$, we have either $\theta_s L = 0$ or $\theta_s L$ has L in the top, cf. Corollary 4. Assume that N has a zero element on the diagonal. From [Zi, Lemma 6.4] (see also Corollary 4) it follows that $\theta_s L = 0$ for some simple object $L \in \overline{\mathbf{M}}(\mathbf{i})$. Then, on the one hand, $\theta_{sts} \theta_s L = 0$, but, on the other hand, $\theta_{sts} \theta_s \cong \theta_{sts} \oplus \theta_{sts}$. Therefore $\theta_{sts} L = 0$. This implies that the matrix M has a zero row and thus is not positive, a contradiction. Therefore all diagonal elements in N are equal to 2. Taking (3) into account, it follows that $y = 0$. \square

4.4. Combinatorial restrictions on matrices. Let \mathbf{M} be a faithful simple transitive 2-representation of \mathcal{Q}_5 , P_1 and P_2 be two non-isomorphic indecomposable objects in $\mathbf{M}(\mathbf{i})$ and L_1 and L_2 the corresponding simple objects in $\overline{\mathbf{M}}(\mathbf{i})$. Consider the transitive module $V = \mathbb{C} \otimes_{\mathbb{Z}} [\mathbf{M}(\mathbf{i})]$ over the positively based algebra A . In this section we determine what the combinatorial possibilities are for the matrices $M_s := (\theta_s)$ and $M_{sts} := (\theta_{sts})$.

Lemma 7. *Up to swapping P_1 and P_2 , the possibilities for the pair (M_s, M_{sts}) are:*

- (a) $\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix} \right),$
- (b) $\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix} \right),$
- (c) $\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix} \right),$
- (d) $\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix} \right).$

Proof. Similarly to [Zi, Lemma 6.4], M_s is equal to twice the identity matrix (it is a rank-two square matrix with 2's on the diagonal satisfying $M_s^2 = 2M_s$). So we only need to determine M_{sts} . Write

$$M_{sts} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $\underline{sts} \cdot \underline{sts} = 2\underline{sts} + 2\underline{s}$, given by (3), is equivalent to the following system of equations:

$$(4) \quad \begin{cases} a^2 + bc &= 2a + 4 \\ (a + d)b &= 2b \\ (a + d)c &= 2c \\ cb + d^2 &= 2d + 4 \end{cases}$$

As $M_s + M_{sts}$ is positive, we have $c \neq 0$ and $b \neq 0$. This implies $a + d = 2$. Hence, up to swapping of P_1 and P_2 , we have $(a, d) = (1, 1)$ or $(a, d) = (0, 2)$.

If $(a, d) = (1, 1)$, then (4) is equivalent to $bc = 5$. Up to swapping of P_1 and P_2 , this gives (a). If $(a, d) = (0, 2)$, then (4) is equivalent to $bc = 4$. Note that in this case we cannot swap P_1 and P_2 anymore, as this will affect the pair (a, d) . Therefore we get all the remaining cases (b), (c) and (d). \square

4.5. Ruling out the case of Lemma 7(a). Here we continue to work in the setup of the previous subsection.

Lemma 8. *For a faithful simple transitive 2-representation of \mathcal{Q}_5 , the case of Lemma 7(a) is not possible.*

Proof. Consider the additive closure of $\theta_s L_1$ and $\theta_{sts} L_1$. It is stable under the action of \mathcal{Q}_5 . The usual argument for θ_s (cf. Corollary 4) implies that, for $i = 1, 2$, the module $\theta_s L_i$ has length two with simple top and socle isomorphic to L_i (see, for example, proof of [MM5, Proposition 22] or Subsection 3.4). Because of the form of the matrix $[[\theta_{sts}]]$ which is transposed to M_{sts} , the module $\theta_{sts} L_1$ is either $L_1 \oplus L_2$ or is uniserial with simple top L_2 and socle L_1 , or vice versa. Applying θ_s , we, on the one hand, should get $\theta_{sts} L_1$ doubled, as $\theta_s \theta_{sts} = \theta_{sts} \oplus \theta_{sts}$. On the other hand, the resulting module must contain a self-extension of both L_1 and L_2 , a contradiction. \square

4.6. Ruling out the case of Lemma 7(d). Here we continue to work in the setup of Subsection 4.4.

Lemma 9. *For a faithful simple transitive 2-representation of \mathcal{Q}_5 , the case of Lemma 7(d) is not possible.*

Proof. Consider the additive closure of $\theta_s L_1$ and $\theta_{sts} L_1$. It is stable under the action of \mathcal{Q}_5 . The usual argument for θ_s (cf. Corollary 4) implies that $\theta_s L_i$ has length two with simple top and socle isomorphic to L_i , for $i = 1, 2$. Because of the form of the matrix $[[\theta_{sts}]]$, the module $\theta_{sts} L_1$ is L_2 . Applying θ_s , we, on the one hand, should get $L_2 \oplus L_2$, as $\theta_s \theta_{sts} = \theta_{sts} \oplus \theta_{sts}$. On the other hand, the resulting module $\theta_s L_2$ is indecomposable as it has simple top, a contradiction. \square

4.7. Ruling out the case of Lemma 7(b). Here we continue to work in the setup of Subsection 4.4.

Lemma 10. *For a faithful simple transitive 2-representation of \mathcal{Q}_5 , the case of Lemma 7(b) is not possible.*

Proof. Consider the additive closure of $\theta_s L_2$ and $\theta_{sts} L_2$. It is stable under the action of \mathcal{Q}_5 . As usual, for $i = 1, 2$, the object $\theta_s L_i$ has length two with simple top and socle isomorphic to L_i . Because of the form of the matrix $[\![\theta_{sts}]\!]$, the module $\theta_{sts} L_2$ has, as subquotients, L_1 (with multiplicity one) and L_2 (with multiplicity two). Similarly to the proof of Lemma 8, the module $\theta_{sts} L_2$ cannot be semi-simple.

As L_1 has multiplicity one in $\theta_{sts} L_2$ and all other simple subquotients are isomorphic to L_2 , either socle or top of $\theta_{sts} L_2$ contains L_2 . By adjunction, we have

$$\mathrm{Hom}(\theta_{sts} L_2, L_2) \cong \mathrm{Hom}(L_2, \theta_{sts} L_2),$$

which implies that L_2 must appear in both, top and socle of $\theta_{sts} L_2$. Now, $\theta_{sts} L_1$ has only L_2 as composition subquotients. Therefore, by adjunction,

$$\mathrm{Hom}(\theta_{sts} L_2, L_1) \cong \mathrm{Hom}(L_2, \theta_{sts} L_1) \neq 0$$

and therefore L_1 is in the top of $\theta_{sts} L_2$. A similar argument gives that L_1 is in the socle of $\theta_{sts} L_2$. As L_1 has multiplicity one in $\theta_{sts} L_2$, it must be a direct summand. Now, applying θ_s , we, on the one hand, should get $L_1 \oplus L_1$, as $\theta_s \theta_{sts} = \theta_{sts} \oplus \theta_{sts}$. On the other hand, in the resulting module, the two subquotients L_1 must be glued into an indecomposable direct summand $\theta_s L_1$, a contradiction. \square

4.8. Both θ_s and θ_{sts} map simples to projectives. Here we continue to work in the setup of Subsection 4.4.

Lemma 11. *For any simple object $L \in \overline{\mathbf{M}}(\mathbf{i})$, both $\theta_s L$ and $\theta_{sts} L$ are projective objects.*

Proof. Combining Lemmata 8, 9 and 10, we know that the pair (M_s, M_{sts}) is given by Lemma 7(c). Let

$$\mathbf{A} : \quad P_1^{\oplus a_1} \oplus P_2^{\oplus a_2} \xrightarrow{\alpha} P_1^{\oplus b_1} \oplus P_2^{\oplus b_2}$$

be a minimal projective presentation of $\theta_s L$ and

$$\mathbf{B} : \quad P_1^{\oplus c_1} \oplus P_2^{\oplus c_2} \xrightarrow{\beta} P_1^{\oplus d_1} \oplus P_2^{\oplus d_2}$$

be a minimal projective presentation of $\theta_{sts} L$. As θ_s is self-adjoint, it maps projective resolutions to projective resolutions. From (2) and the explicit matrix for M_s we obtain that θ_s doubles both θ_s , θ_{sts} and all projective modules. Hence θ_s sends \mathbf{A} to $\mathbf{A} \oplus \mathbf{A}$ and \mathbf{B} to $\mathbf{B} \oplus \mathbf{B}$.

As θ_{sts} is self-adjoint, it maps projective resolutions to projective resolutions (however, we do not know whether it sends minimal resolutions to minimal resolutions). Applying θ_{sts} to \mathbf{A} and \mathbf{B} and using (2) and the explicit form of M_{sts} produces the following two systems of inequalities:

$$\left\{ \begin{array}{lcl} 2b_2 & \geq & 2d_1 \\ 2b_1 + 2b_2 & \geq & 2d_2 \\ 2d_2 & \geq & 2d_1 + 2b_1 \\ 2d_1 + 2d_2 & \geq & 2d_2 + 2b_2 \end{array} \right. \quad \left\{ \begin{array}{lcl} 2a_2 & \geq & 2c_1 \\ 2a_1 + 2a_2 & \geq & 2c_2 \\ 2c_2 & \geq & 2c_1 + 2a_1 \\ 2c_1 + 2c_2 & \geq & 2c_2 + 2a_2 \end{array} \right.$$

These inequalities imply the equalities $d_1 = b_2$, $d_2 = b_1 + b_2$, $c_1 = a_2$ and also $c_2 = a_1 + a_2$. Therefore θ_{sts} maps \mathbf{A} to $\mathbf{B} \oplus \mathbf{B}$ and \mathbf{B} to $\mathbf{A} \oplus \mathbf{A} \oplus \mathbf{B} \oplus \mathbf{B}$.

The above proves that the ideal generated by α and β is stable under the action of \mathcal{Q}_5 . Since \mathbf{M} is simple transitive, this ideal therefore has to be zero, so $\alpha = \beta = 0$, cf. [MM5, Lemma 12]. This completes the proof. \square

4.9. Proof of Theorem 5. Here we continue to work under the assumptions of Subsection 4.4. Combining Lemmata 8, 9 and 10, we know that the pair (M_s, M_{sts}) is given by Lemma 7(c). Consider the additive closure of $\theta_s L_1$ and $\theta_{sts} L_1$. Both these objects are projective and the additive closure is stable under the action of \mathcal{Q}_5 . The module $\theta_s L_1$ is an indecomposable module of length two with simple top and socle isomorphic to L_1 , by the usual argument (cf. Corollary 4). Therefore $\theta_s L_1 \cong P_1$.

The module $\theta_{sts} L_1$ has length two and cannot be semisimple by the argument in the proof of Lemma 8. Therefore it is an indecomposable module of length two with simple top and socle isomorphic to L_2 . Therefore $\theta_{sts} L_1 \cong P_2$. This implies that the Cartan matrix of \mathbf{M} is

$$(5) \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Mapping $\mathbb{1}_i$ to L_1 extends to a strict 2-natural transformation $\Phi : \mathbf{P}_i \rightarrow \mathbf{M}$.

Consider now the cell 2-representation $\mathbf{C}_{\mathcal{L}_s}$. It is a faithful simple transitive 2-representation of \mathcal{Q}_5 . Hence the above shows that (5) is also the Cartan matrix of $\mathbf{C}_{\mathcal{L}_s}$. From the uniqueness of a maximal left ideal in the construction of cell 2-representations, it follows that Φ factors through $\mathbf{C}_{\mathcal{L}_s}$. Comparing the Cartan matrices, we see that the induced 2-natural transformation from $\mathbf{C}_{\mathcal{L}_s}$ to \mathbf{M} is an equivalence.

5. SIMPLE TRANSITIVE 2-REPRESENTATIONS OF \mathcal{Q}_4

5.1. The result. Our main result for the 2-category \mathcal{Q}_4 is the following:

Theorem 12. *The 2-category \mathcal{Q}_4 has three equivalence classes of simple transitive 2-representations, namely, the cell 2-representations $\mathbf{C}_{\mathcal{L}_e}$ and $\mathbf{C}_{\mathcal{L}_s}$ together with the simple transitive 2-representation \mathbf{N} constructed in Subsection 5.8.*

5.2. The algebra $[\mathcal{Q}_4](i, i)$. We identify the ring $R = [\mathcal{Q}_4](i, i)$ with the corresponding subquotient of $\mathbb{Z}D_{2,4}$. With this identification, the ring R has basis $\{\underline{e}, \underline{s}, \underline{sts}\}$ and the following multiplication table of $x \cdot y$:

$$(6) \quad \begin{array}{c|c|c|c} x \backslash y & \underline{e} & \underline{s} & \underline{sts} \\ \hline \underline{e} & \underline{e} & \underline{s} & \underline{sts} \\ \hline \underline{s} & \underline{s} & 2\underline{s} & 2\underline{sts} \\ \hline \underline{sts} & \underline{sts} & 2\underline{sts} & 2\underline{s} \end{array}$$

The \mathbb{C} -algebra $A := \mathbb{C} \otimes_{\mathbb{Z}} R$ is commutative and split semisimple. The three 1-dimensional representations of A are given by the following table which describes the action of the basis elements in any fixed basis:

$$(7) \quad \begin{array}{c|c|c|c} & V_1 & V_2 & V_3 \\ \hline \underline{e} & 1 & 1 & 1 \\ \hline \underline{s} & 0 & 2 & 2 \\ \hline \underline{sts} & 0 & -2 & 2 \end{array}$$

Cells in A have exactly the same combinatorics as for \mathcal{Q}_4 . For the cell \mathcal{L}_e , the special A -module, in the sense of [KM2], is V_1 ; for the cell \mathcal{L}_s , the special A -module is V_3 .

5.3. Reduction to ranks one and two. Let $\hat{\mathcal{Q}}_4$ be the quotient of \mathcal{Q}_4 by the 2-ideal generated by 1-morphisms in \mathcal{J} . Then the only surviving indecomposable 1-morphism in $\hat{\mathcal{Q}}_5$ is the identity 1-morphism, up to isomorphism. Therefore each simple transitive 2-representation of $\hat{\mathcal{Q}}_4$ is a cell 2-representation by [MM5, Theorem 18].

Let \mathbf{M} be a simple transitive 2-representation of \mathcal{Q}_4 . Recall that \mathcal{Q}_4 is \mathcal{J} -simple. Therefore, if \mathbf{M} is not faithful, it factors through $\hat{\mathcal{Q}}_4$. Consequently, from the previous paragraph we obtain that \mathbf{M} is a cell 2-representation. Therefore, from now on in this section we may assume that \mathbf{M} is faithful.

Lemma 13. *A faithful simple transitive 2-representation \mathbf{M} of \mathcal{Q}_4 either has rank two and decategorifies to $V_2 \oplus V_3$ or it has rank one and decategorifies to V_3 .*

Proof. Consider the transitive module $V = \mathbb{C} \otimes_{\mathbb{Z}} [\mathbf{M}(\mathbf{i})]$ over the positively based algebra A , see Subsection 2.3. As \mathbf{M} is faithful, either V_2 or V_3 must appear as a subquotient of V . As V_3 is special, it has multiplicity one. Since V_2 is not special, it follows that

$$V \cong V_3 \oplus V_2^{\oplus x} \oplus V_1^{\oplus y}.$$

Similarly to the proof of Lemma 6 one shows that $y = 0$. As \underline{sts} must have non-negative trace, we have $x \leq 1$. The claim follows. \square

5.4. Combinatorial restrictions on matrices. Let \mathbf{M} be a faithful simple transitive 2-representation of \mathcal{Q}_4 . Let P_1 or P_1 and P_2 be non-isomorphic indecomposable objects in $\mathbf{M}(\mathbf{i})$ and L_1 or L_1 and L_2 the corresponding simple objects in $\overline{\mathbf{M}}(\mathbf{i})$. Consider the transitive module $V = \mathbb{C} \otimes_{\mathbb{Z}} [\mathbf{M}(\mathbf{i})]$ over the positively based algebra A . In this section we determine what the combinatorial possibilities for the matrices $M_s := \langle \theta_s \rangle$ and $M_{sts} := \langle \theta_s \rangle$ are.

Lemma 14. *Up to swapping P_1 and P_2 , the only possibilities for the pair (M_s, M_{sts}) are:*

- (a) $((2), (2))$,
- (b) $\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \right)$,
- (c) $\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \right)$.

Proof. If \mathbf{M} has rank one, the possibility (a) follows combining Lemma 13 and (7).

If \mathbf{M} has rank two, then, as in Lemma 7, M_s is equal to twice the identity matrix, so we only need to determine M_{sts} . Write

$$M_{sts} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $\underline{sts} \cdot \underline{sts} = 2\underline{s}$, given by (6), is equivalent to the following system of equations:

$$\begin{cases} a^2 + bc &= 4 \\ (a + d)b &= 0 \\ (a + d)c &= 0 \\ cb + d^2 &= 4 \end{cases}$$

As $M_s + M_{sts}$ is positive, we have $c \neq 0$ and $b \neq 0$. This implies $a + d = 0$ and thus $a = 0 = d$ and $bc = 4$. Up to swapping of P_1 and P_2 , this gives possibilities (b) and (c). \square

5.5. Ruling out the case of Lemma 14(c). Here we continue to work in the setup of the previous subsection.

Lemma 15. *For a faithful simple transitive 2-representation of \mathcal{Q}_4 , the case of Lemma 14(c) is not possible.*

Proof. Consider the additive closure of $\theta_s L_2$ and $\theta_{sts} L_2$. It is stable under the action of \mathcal{Q}_4 . The usual argument for θ_s (cf. Corollary 4) implies that $\theta_s L_i$ has length two with simple top and socle isomorphic to L_i , for $i = 1, 2$. Because of the form of the matrix $[\![\theta_{sts}]\!]$ which is transposed to M_{sts} , we have $\theta_{sts} L_2 \cong L_1$. Now, on the one hand, $\theta_s \theta_{sts} L_2 \cong L_1 \oplus L_1$, as $\theta_s \theta_{sts} = \theta_{sts} \oplus \theta_{sts}$. On the other hand, the usual argument for θ_s (cf. Corollary 4) implies that $\theta_s L_1$ has length two with simple top and socle isomorphic to L_1 , a contradiction. \square

5.6. Both θ_s and θ_{sts} map simples to projectives. Here we continue to work in the setup of Subsection 5.4.

Lemma 16. *For any simple object $L \in \overline{\mathbf{M}}(\mathbf{i})$, both $\theta_s L$ and $\theta_{sts} L$ are projective objects.*

Proof. Combining Lemmata 14 and 15, we know that the pair (M_s, M_{sts}) is given by Lemma 14(a) or by Lemma 14(b). In the rank one case given by Lemma 14(a) it is clear that both θ_s and θ_{sts} double both $\theta_s L$, $\theta_{sts} L$ and all projectives. Therefore they send minimal projective resolutions to minimal projective resolutions and the claim follows by similar arguments as in Lemma 11.

In the rank two case given by Lemma 14(b), θ_s doubles everything; while θ_{sts} doubles and swaps indices. Again, it follows easily that both θ_s and θ_{sts} send minimal projective resolutions to minimal projective resolutions and the claim follows by similar arguments as in Lemma 11. \square

5.7. Some evidence for the existence of an additional simple transitive 2-representation of \mathcal{Q}_4 . The ring $[\mathcal{Q}_6](\mathbf{i}, \mathbf{i})$ has basis $\{\underline{e}, \underline{s}, \underline{sts}, \underline{ststs}\}$ and the following multiplication table of $x \cdot y$:

$x \setminus y$	\underline{e}	\underline{s}	\underline{sts}	\underline{ststs}
\underline{e}	\underline{e}	\underline{s}	\underline{sts}	\underline{ststs}
\underline{s}	\underline{s}	$2\underline{s}$	$2\underline{sts}$	$2\underline{ststs}$
\underline{sts}	\underline{sts}	$2\underline{sts}$	$2\underline{s} + 2\underline{sts} + 2\underline{ststs}$	$2\underline{sts}$
\underline{ststs}	\underline{ststs}	$2\underline{ststs}$	$2\underline{sts}$	$2\underline{s}$

Comparing this with (6) suggests the possibility of a connection between \mathcal{Q}_4 and the 2-subcategory $\tilde{\mathcal{Q}}_6$ of \mathcal{Q}_6 generated by θ_s and θ_{ststs} (however, establishing such a connection explicitly might be very hard).

Now we can consider the restriction of the cell 2-representation $\mathbf{C}_{\mathcal{L}_s}$ of \mathcal{Q}_6 to $\tilde{\mathcal{Q}}_6$. This restriction is not transitive, so we can consider its weak Jordan-Hölder series in the sense of [MM5, Section 4]. From the above multiplication table, we see that $\underline{ststs} \cdot \underline{sts} = 2\underline{sts}$. This means that there is a simple transitive subquotient in this weak Jordan-Hölder series, for which the pair $([\![\theta_s]\!], [\![\theta_{ststs}]\!])$ of matrices has the form $((2), (2))$. This suggests that a similar thing should also exist for \mathcal{Q}_4 .

5.8. An additional simple transitive 2-representation of \mathcal{Q}_4 . The aim of this subsection is to construct a simple transitive 2-representation of \mathcal{Q}_4 which is not equivalent to a cell 2-representation. This is done in several steps:

- We start with the cell 2-representation \mathbf{Q} of \mathcal{Q}_4 which corresponds to the cell \mathcal{J} . The underlying category of this 2-representation has two isomorphism classes of indecomposable objects. We observe that there is a 2-representation \mathbf{K} of \mathcal{Q}_4 which is equivalent to \mathbf{Q} and which has a non-trivial automorphism swapping the indecomposable objects.
- Next, we modify \mathbf{K} to another equivalent 2-representation \mathbf{L} which has the same kind of automorphism that, additionally, is a strict involution, i.e. squares to the identity functor. At the intermediate stage we modify \mathbf{K} to another equivalent 2-representation which has a coherent $\mathbb{Z}/2\mathbb{Z}$ -action (we use the terminology of [Se, Pages 135-136]). This involves a subtle chase of certain modifications for the cell 2-representation, which is rather technical and occupies a major part of this subsection.
- Finally, we use the orbit category construction (see [CM] and Subsection 5.9 for details) to produce a “quotient” of \mathbf{L} which turns out to be a simple transitive 2-representation of \mathcal{Q}_4 whose underlying category has one isomorphism class of indecomposable objects.

So, let us now do the work. Consider the cell 2-representation $\mathbf{Q} := \mathbf{C}_{\mathcal{L}_s}$ of \mathcal{Q}_4 and its abelianization $\overline{\mathbf{Q}}$. Let P_s and P_{sts} be representatives of the isomorphism classes of the indecomposable projective objects in $\overline{\mathbf{Q}}(\mathbf{i})$ and L_s and L_{sts} be the respective simple tops. Let B denote the basic underlying algebra of $\overline{\mathbf{Q}}(\mathbf{i})$ and f_s and f_{sts} denote pairwise orthogonal primitive idempotents of B corresponding to P_s and P_{sts} , respectively. The matrices of the action of θ_s and θ_{sts} are given by Lemma 14(b). As \mathbf{Q} is simple transitive, both θ_s and θ_{sts} send simple objects in $\overline{\mathbf{Q}}(\mathbf{i})$ to projective objects in $\overline{\mathbf{Q}}(\mathbf{i})$, by Lemma 16. As \mathcal{Q}_4 is fiat, from [MM5, Lemma 13] it follows that both θ_s and θ_{sts} act as projective endofunctors of $\overline{\mathbf{Q}}(\mathbf{i})$.

Recall that D is the algebra of dual numbers introduced in Subsection 3.3. Taking into account the results of Subsection 3.4, from the matrix M_s we can deduce that $B \cong D \oplus D$ and $f_s + f_{sts} = 1$. The action of θ_s is given by tensoring with the B - B -bimodule $(Bf_s \otimes f_s B) \oplus (Bf_{sts} \otimes f_{sts} B)$ while the action of θ_{sts} is, similarly, given by tensoring with the B - B -bimodule $(Bf_{sts} \otimes f_s B) \oplus (Bf_s \otimes f_{sts} B)$.

Our first intermediate goal is to construct a strict 2-natural automorphism of \mathbf{Q} which swaps the isomorphism classes of indecomposable objects in $\mathbf{Q}(\mathbf{i})$. We cannot do it directly and instead construct a different, but equivalent, 2-representation \mathbf{K} where such a strict 2-natural automorphism is easy to find.

Set $\mathbf{Q}^{(0)} := \mathbf{Q}$ and let $\mathbf{Q}^{(1)}$ denote the 2-representation of \mathcal{Q}_4 given by the action of \mathcal{Q}_4 on the category of projective objects in $\overline{\mathbf{Q}}^{(0)}(\mathbf{i})$. Recursively, for $k \geq 1$, define $\mathbf{Q}^{(k)}$ as the 2-representation of \mathcal{Q}_4 given by the action of \mathcal{Q}_4 on the category of projective objects in $\overline{\mathbf{Q}}^{(k-1)}(\mathbf{i})$. For every $k \geq 0$, we have a strict 2-natural transformation $\Lambda_k : \mathbf{Q}^{(k-1)} \rightarrow \mathbf{Q}^{(k)}$ which sends an object X to the diagram $0 \rightarrow X$ and a morphism $\alpha : X \rightarrow X'$ to the diagram

$$\begin{array}{ccc} 0 & \xrightarrow{\quad} & X \\ \downarrow & & \downarrow \alpha \\ 0 & \xrightarrow{\quad} & X'. \end{array}$$

Clearly, each such Λ_k is an equivalence.

Denote by \mathbf{K} the inductive limit of the directed system

$$(8) \quad \mathbf{Q}^{(0)} \xrightarrow{\Lambda_0} \mathbf{Q}^{(1)} \xrightarrow{\Lambda_1} \mathbf{Q}^{(2)} \xrightarrow{\Lambda_2} \dots$$

Then \mathbf{K} is a 2-representation of \mathcal{Q}_4 which is equivalent to \mathbf{Q} .

Lemma 17. *There is a strict 2-natural transformation $\Psi : \mathbf{K} \rightarrow \mathbf{K}$ which is an equivalence and which swaps the isomorphism classes of the indecomposable projective objects.*

Proof. Consider the unique strict 2-natural transformation $\Phi : \mathbf{P}_i \rightarrow \overline{\mathbf{Q}}$ which sends $\mathbb{1}_i$ to L_{sts} . By the above discussion, both $\theta_s L_{sts}$ and $\theta_{sts} L_{sts}$ are indecomposable objects in $\mathbf{Q}^{(1)}(i)$. Using similar arguments as in Subsection 4.9, it follows that Φ factors through $\mathbf{C}_{\mathcal{L}_s}$ and, therefore, gives rise to a strict equivalence $\Phi^{(0)} : \mathbf{Q}^{(0)} \rightarrow \mathbf{Q}^{(1)}$. Applying abelianization, for every $k \geq 0$, we obtain a strict equivalence $\Phi^{(k)} : \mathbf{Q}^{(k)} \rightarrow \mathbf{Q}^{(k+1)}$, which is, by construction, compatible with (8). Now we can take Ψ as the inductive limit of $\Phi^{(k)}$. \square

Consider a new finitary 2-representation \mathbf{K}' of \mathcal{Q}_4 defined as follows:

- Objects of $\mathbf{K}'(i)$ are sequences $(X_n, \alpha_n)_{n \in \mathbb{Z}}$, where X_n is an object in $\mathbf{K}(i)$ and $\alpha_n : \Psi(X_n) \rightarrow X_{n+1}$ an isomorphism in $\mathbf{K}(i)$, for all $n \in \mathbb{Z}$.
- Morphisms in $\mathbf{K}'(i)$ from $(X_n, \alpha_n)_{n \in \mathbb{Z}}$ to $(Y_n, \beta_n)_{n \in \mathbb{Z}}$ are sequences of morphisms $f_n : X_n \rightarrow Y_n$ in $\mathbf{K}(i)$ such that

$$\begin{array}{ccc} \Psi(X_n) & \xrightarrow{\alpha_n} & X_{n+1} \\ \Psi(f_n) \downarrow & & \downarrow f_{n+1} \\ \Psi(Y_n) & \xrightarrow{\beta_n} & Y_{n+1} \end{array}$$

commutes for all $n \in \mathbb{Z}$.

- The action of \mathcal{Q}_4 on $\mathbf{K}'(i)$ is inherited from the action of \mathcal{Q}_4 on $\mathbf{K}(i)$ component-wise.

The construction of $\mathbf{K}'(i)$ from $\mathbf{K}(i)$ is the standard construction which turns a category with an autoequivalence (in our case Ψ) into an equivalent category with an automorphism, cf. [Ke, BL].

We have the strict 2-natural transformation $\Pi : \mathbf{K}' \rightarrow \mathbf{K}$ given by projection onto the zero component of a sequence. This Π is an equivalence, by construction. We also have a strict 2-natural transformation $\Psi' : \mathbf{K}' \rightarrow \mathbf{K}'$ given by shifting the entries of the sequences by one, that is, sending $(X_n, \alpha_n)_{n \in \mathbb{Z}}$ to $(X_{n+1}, \alpha_{n+1})_{n \in \mathbb{Z}}$, with the similar obvious action on morphisms. Note that $\Psi' : \mathbf{K}'(i) \rightarrow \mathbf{K}'(i)$ is an automorphism. Similarly to Ψ , the functor Ψ' swaps the isomorphism classes of indecomposable objects in $\mathbf{K}'(i)$. As Ψ^2 is isomorphic to the identity functor on $\mathbf{K}(i)$, it follows, by construction, that $(\Psi')^2$ is isomorphic to the identity functor on $\mathbf{K}'(i)$. However, we need the following stronger statement.

Lemma 18. *Let $\text{Id} : \mathbf{K}' \rightarrow \mathbf{K}'$ denote the identity 2-natural transformation.*

- (i) *There is an invertible modification $\eta : \text{Id} \rightarrow (\Psi')^2$.*
- (ii) *For any η as in (i), we have $\text{id}_{(\Psi')^2} \circ_0 \eta = \eta \circ_0 \text{id}_{(\Psi')^2}$.*
- (iii) *For any $\eta' \in \text{Hom}_{\mathcal{Q}_4\text{-mod}}(\text{Id}, (\Psi')^2)$, we have $\text{id}_{(\Psi')^2} \circ_0 \eta' = \eta' \circ_0 \text{id}_{(\Psi')^2}$.*

Proof. Let X_1 and X_2 be representatives of the different isomorphism classes of indecomposable objects in $\mathbf{K}'(\mathbf{i})$. Fix an isomorphism $\eta_{X_1} : X_1 \rightarrow (\Psi')^2(X_1)$. Note that, after the abelianization of \mathbf{K}' , the object X_1 becomes isomorphic to $\theta_s L_1$, where L_1 is the simple top of the indecomposable projective object $0 \rightarrow X_1$. As $\theta_{sts} \circ \theta_s \cong \theta_{sts} \oplus \theta_{sts}$, and $(\Psi')^2$ is a strict 2-natural transformation, it follows that the morphism $\theta_{sts}(\eta_{X_1})$ can be written in the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

for some isomorphism $\alpha : \theta_{sts} L_1 \rightarrow (\Psi')^2(\theta_{sts} L_1)$. We may now define η_{X_2} as the map induced from α via a fixed isomorphism $X_2 \cong \theta_{sts} L_1$. This uniquely determines a bijective natural transformation $\eta : \text{Id} \rightarrow (\Psi')^2$.

As $\theta_{sts} \circ \theta_s \cong \theta_s \oplus \theta_s$, from the above construction it follows that

$$(9) \quad \text{id}_{\theta_{sts}} \circ \eta = \eta \circ \text{id}_{\theta_{sts}}.$$

Note that the 2-category \mathcal{Q}_4 is monoidally generated by θ_{sts} . Therefore (9) implies that the η constructed above is, in fact, a modification. Claim (i) follows.

Because of the naturality of η , we have the commutative diagram

$$\begin{array}{ccc} (\Psi')^{-2} & \xrightarrow{\eta \circ \text{id}_{(\Psi')^{-2}}} & \text{Id} \\ \eta \circ \text{id}_{(\Psi')^{-2}} \downarrow & & \downarrow \text{id}_{(\Psi')^2} \circ \eta \circ \text{id}_{(\Psi')^{-2}} \\ \text{Id} & \xrightarrow{\eta} & (\Psi')^2 \end{array}$$

As $\eta \circ \text{id}_{(\Psi')^{-2}}$ is invertible, claim (ii) follows. As invertible elements inside the space $\text{Hom}_{\mathcal{Q}_4\text{-mod}}(\text{Id}, (\Psi')^2)$ exist by claim (i), they also span this space (by the usual argument that invertible modifications form a Zariski open set which is non-empty by claim (i)). Therefore claim (iii) follows from claim (ii) by linearity. \square

The next statement basically says that there is a coherent action of the group $\mathbb{Z}/2\mathbb{Z}$ on \mathbf{K}' .

Proposition 19. *There is an invertible modification $\eta : \text{Id} \rightarrow (\Psi')^2$ for which we have the equality $\text{id}_{\Psi'} \circ \eta = \eta \circ \text{id}_{\Psi'}$.*

Proof. Consider the space $\text{Hom}(\text{Id}, (\Psi')^2)$ of all natural transformations from Id to $(\Psi')^2$, as endofunctors of $\mathbf{K}'(\mathbf{i})$. As Id and $(\Psi')^2$ are isomorphic, the space $\text{Hom}(\text{Id}, (\Psi')^2)$ is isomorphic, by fixing any isomorphism between Id and $(\Psi')^2$, to the space $\text{End}(\text{Id})$ of natural endomorphisms of the identity functor on $\mathbf{K}'(\mathbf{i})$. We have the usual isomorphism $\text{End}(\text{Id}) \cong B$, since B is commutative. In particular, $\text{End}(\text{Id})$ has dimension four.

The vector space $\text{Hom}_{\mathcal{Q}_4\text{-mod}}(\text{Id}, (\Psi')^2)$ of modifications is a subspace in the space $\text{Hom}(\text{Id}, (\Psi')^2)$. An element $\zeta \in \text{Hom}(\text{Id}, (\Psi')^2)$ is uniquely determined by its values on any pair X_1 and X_2 of representatives of the two isomorphism classes of indecomposable objects in $\mathbf{K}'(\mathbf{i})$. The 2-category \mathcal{Q}_4 is monoidally generated by θ_{sts} . Therefore, as $\theta_{sts} X_1 \cong X_2 \oplus X_2$, the axiom $\zeta \circ \text{id}_{\theta_{sts}} = \text{id}_{\theta_{sts}} \circ \zeta$ for modifications implies that an element $\zeta \in \text{Hom}_{\mathcal{Q}_4\text{-mod}}(\text{Id}, (\Psi')^2)$ is uniquely determined by its value on X_1 . Consequently, $\text{Hom}_{\mathcal{Q}_4\text{-mod}}(\text{Id}, (\Psi')^2)$ has dimension two.

The map

$$(10) \quad \zeta \mapsto \text{id}_{\Psi'} \circ \zeta \circ \text{id}_{\Psi'}$$

is an invertible linear transformation of the space $\text{Hom}(\text{Id}, (\Psi')^2)$ which preserves $\text{Hom}_{\mathcal{Q}_{4\text{-mod}}}(\text{Id}, (\Psi')^2)$. We denote this transformation by T . From Lemma 18(iii) we know that T^2 is the identity map, when restricted to $\text{Hom}_{\mathcal{Q}_{4\text{-mod}}}(\text{Id}, (\Psi')^2)$. Consequently, T is diagonalizable (in the restriction) with eigenvalues 1 and -1 .

The algebra D is positively graded in the usual way (the degree of x is two), which also makes B into a positively graded algebra. The bimodules which represent the actions of θ_s and θ_{sts} are gradeable and all our constructions above are gradeable as well. Consequently, $\text{Hom}(\text{Id}, (\Psi')^2)$ inherits a grading from B and $\text{Hom}_{\mathcal{Q}_{4\text{-mod}}}(\text{Id}, (\Psi')^2)$ is a graded subspace of $\text{Hom}(\text{Id}, (\Psi')^2)$ isomorphic to D (as a graded space). The map (10) is homogeneous of degree zero, which implies that both homogeneous elements in $\text{Hom}_{\mathcal{Q}_{4\text{-mod}}}(\text{Id}, (\Psi')^2)$ are eigenvectors for the linear map (10).

Let $\eta \in \text{Hom}_{\mathcal{Q}_{4\text{-mod}}}(\text{Id}, (\Psi')^2)$ be a non-zero homogeneous element of degree zero. Then η is, obviously, invertible. By the above, η is an eigenvector for the map (10). To complete the proof, we only need to check that the corresponding eigenvalue is 1 and not -1 .

To check the eigenvalue, we will use a description of both η and Ψ' in terms of B - B -bimodules which is explicit enough to see that the transformation (10), when applied to η , cannot introduce any signs. For this we need to recall the construction of Ψ' which is based on the construction of Ψ (cf. [MM1, Subsection 4.6]). Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be a basis in the linear space of degree two 2-endomorphisms of θ_{sts} . Then the module L_{sts} in $\overline{\mathbf{Q}}(\mathbf{i})$, which was used to define Ψ , has a presentation of the form

$$\theta_{sts}^{\oplus k} \xrightarrow{\alpha} \theta_{sts},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$. For objects $\theta \in \mathbf{C}_{\mathcal{L}_s}$, the functor Ψ is given by

$$\theta \mapsto \theta \theta_{sts}^{\oplus k} \xrightarrow{\theta(\alpha)} \theta \theta_{sts},$$

by construction. The 2-representation \mathbf{K}' is 2-faithful because of \mathcal{J} -simplicity of \mathcal{Q}_5 , see Subsection 3.2. Therefore, using the connection to $B\text{-mod}$ as explained in the beginning of this subsection, the above can be written in terms of explicit B - B -bimodules and bimodule maps which represent the action of θ_s and θ_{sts} . Let us denote by Q_s and Q_{sts} the B - B -bimodules which represent the actions of θ_s and θ_{sts} , respectively.

We can also consider a similar presentation

$$\theta_s^{\oplus m} \xrightarrow{\beta} \theta_s,$$

for L_S , where $\beta = (\beta_1, \beta_2, \dots, \beta_m)$ with $\beta_1, \beta_2, \dots, \beta_m$ being a basis in the linear space of degree two 2-endomorphisms of θ_s . Then the functor

$$\theta \mapsto \theta \theta_s^{\oplus m} \xrightarrow{\theta(\beta)} \theta \theta_s$$

is isomorphic to the identity functor on the cell 2-representation via the isomorphism induced by the multiplication map $\mu : B \otimes_{\mathbb{C}} B \rightarrow B$ (note that Q_s is a direct summand of $B \otimes_{\mathbb{C}} B$). Note that μ does not introduce any additional signs to its arguments.

We have the usual isomorphism $Q_{sts} \otimes_B Q_{sts} \cong Q_s \oplus Q_s \langle -2 \rangle$, where $\langle -2 \rangle$ denotes the degree shift by 2 in the positive direction (i.e. with the top concentrated in degree two). Composition of the projection $Q_{sts} \otimes_B Q_{sts} \twoheadrightarrow Q_s$ with the multiplication map μ from the previous paragraph gives rise to a natural transformation from Ψ^2 to the identity functor. In degree zero, the projection $Q_{sts} \otimes_B Q_{sts} \twoheadrightarrow Q_s$ is given in terms of contracting to \mathbb{C} some middle tensor factors, surrounded by $\otimes_{\mathbb{C}}$,

of a tensor monomial. Note that this does not introduce any additional signs. By construction, the induced 2-natural transformation is a homogeneous modification of degree zero, so we can choose η such that it corresponds to this map.

We can take θ_s and θ_{sts} as representatives of isomorphism classes of indecomposable objects in the cell 2-representation. The modification η is uniquely determined by its values on any of these objects. We have to check that $\Psi(\eta) = \eta_\Psi$, while we already know that $\Psi(\eta) = \pm\eta_\Psi$. To do this, we can explicitly write down the bimodules representing the evaluations of $\Psi(\eta)$ and η_Ψ at some object, say θ_s . This gives rather big complexes of bimodules, in which we just need to compare the degree zero parts in position zero. We already know that the degree zero parts are either equal or differ by a sign. However, as explained above, neither the multiplication map nor the projection $Q_{sts} \otimes_B Q_{sts} \rightarrow Q_s$ introduce any new signs in our picture. As we work over \mathbb{C} , that is in characteristic zero, we cannot introduce any signs by adding and multiplying positive elements. This implies that, indeed, $\Psi(\eta) = \eta_\Psi$. The statement of the proposition follows. \square

From now on we fix some invertible modification $\eta : \text{Id} \rightarrow (\Psi')^2$ as given by Proposition 19. Now we use η to replace \mathbf{K}' by an equivalent 2-representation \mathbf{L} in which there is an analogue of Ψ' (see the definition of Θ below) that is, additionally, a strict involution. Define a small category $\mathbf{L}(\mathbf{i})$ as follows:

- objects in $\mathbf{L}(\mathbf{i})$ are all 4-tuples (X, Y, α, β) , where we have $X, Y \in \mathbf{K}'(\mathbf{i})$, while $\alpha : X \rightarrow \Psi'(Y)$ and $\beta : Y \rightarrow \Psi'(X)$ are isomorphisms such that the following conditions are satisfied

$$(11) \quad \begin{cases} \eta_Y^{-1} \circ_1 \Psi'(\alpha) \circ_1 \beta &= \text{id}_Y, \\ \beta \circ_1 \eta_Y^{-1} \circ_1 \Psi'(\alpha) &= \text{id}_{\Psi'(X)}, \\ \eta_X^{-1} \circ_1 \Psi'(\beta) \circ_1 \alpha &= \text{id}_X, \\ \alpha \circ_1 \eta_X^{-1} \circ_1 \Psi'(\beta) &= \text{id}_{\Psi'(Y)}. \end{cases}$$

- morphisms in $\mathbf{L}(\mathbf{i})$ from (X, Y, α, β) to $(X', Y', \alpha', \beta')$ are pairs (ζ, ξ) , where $\zeta : X \rightarrow X'$ and $\xi : Y \rightarrow Y'$ are morphisms in $\mathbf{K}'(\mathbf{i})$ such that the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & \Psi(Y) \\ \zeta \downarrow & & \downarrow \Psi'(\xi) \\ X' & \xrightarrow{\alpha'} & \Psi'(Y') \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \xrightarrow{\beta} & \Psi(X) \\ \xi \downarrow & & \downarrow \Psi'(\zeta) \\ Y' & \xrightarrow{\beta'} & \Psi'(X') \end{array}$$

commute;

- the composition and identity morphisms are the obvious ones.

The category $\mathbf{L}(\mathbf{i})$ comes equipped with an action of \mathcal{Q}_4 , defined component-wise, using the action of \mathcal{Q}_4 on $\mathbf{K}(\mathbf{i})$. This is well-defined as the 2-natural transformation Ψ is strict by Lemma 17 and, moreover, η is a modification. We denote the corresponding 2-representation of \mathcal{Q}_4 by $\mathbf{L}(\mathbf{i})$.

Lemma 20. *Restriction to the first component of a quadruple defines a strict 2-natural transformation $\Upsilon : \mathbf{L} \rightarrow \mathbf{K}'$. This Υ is an equivalence.*

Proof. That the restriction in question is a strict 2-natural transformation is clear by construction. We need to check that it is an equivalence. For this we need to check two things. The first one is the fact that β is uniquely determined by α . and that ξ is uniquely determined by ζ . This follows easily from the definitions and the

equations in (11). The second thing to check is that, given X, Y and α , there is a β such that (11) is satisfied.

By assumptions, we have $X \cong \Psi'(Y)$. The first two equations in (11) just say that β and $\eta_Y^{-1} \circ_1 \Psi'(\alpha)$ are mutual inverses in $\text{Hom}_{\mathbf{L}(\mathbf{i})}(X, \Psi'(Y))$ and $\text{Hom}_{\mathbf{L}(\mathbf{i})}(\Psi'(Y), X)$. Therefore these two equations describe equivalent conditions on α and β . Similarly, the two last equations in (11) describe equivalent conditions on α and β . It remains to check that the first equation is equivalent to the last one. For this we apply Ψ' to the first equation and compare the outcome with the last equation, taking into account that $\Psi'(\beta)$ is an isomorphism. We get

$$(12) \quad \Psi'(\eta_Y^{-1}) \circ_1 \Psi'^2(\alpha) = \alpha \circ_1 \eta_X^{-1}.$$

As $\Psi'(\eta_Y) = \eta_{\Psi'(Y)}$, by our choice of η and Proposition 19, the equality in (12) holds due to naturality of η . The claim of the lemma follows. \square

Define an endofunctor Θ on $\mathbf{L}(\mathbf{i})$ by sending (X, Y, α, β) to (Y, X, β, α) with the obvious action on morphisms. From all symmetries in the definition of $\mathbf{L}(\mathbf{i})$ it follows that Θ is a strict involution and it also strictly commutes with the action of \mathcal{Q}_4 .

Finally, consider the category $\mathbf{N}(\mathbf{i})$ defined as follows:

- $\mathbf{N}(\mathbf{i})$ has the same objects as $\mathbf{L}(\mathbf{i})$,
- morphisms in $\mathbf{N}(\mathbf{i})$ are defined, for objects $X, Y \in \mathbf{L}(\mathbf{i})$, via
$$\text{Hom}_{\mathbf{N}(\mathbf{i})}(X, Y) := \text{Hom}_{\mathbf{L}(\mathbf{i})}(X, Y) \oplus \text{Hom}_{\mathbf{L}(\mathbf{i})}(X, \Theta(Y)),$$
- composition and identity morphisms in $\mathbf{N}(\mathbf{i})$ are induced from those in $\mathbf{L}(\mathbf{i})$ in the obvious way, see [CM, Definition 2.3] for details.

Proposition 21.

- (i) *The category $\mathbf{N}(\mathbf{i})$ is a finitary \mathbb{C} -linear category.*
- (ii) *The category $\mathbf{N}(\mathbf{i})$ is equipped with an action of \mathcal{Q}_4 induced from that on $\mathbf{L}(\mathbf{i})$.*
- (iii) *The obvious functor $\Xi : \mathbf{L}(\mathbf{i}) \rightarrow \mathbf{N}(\mathbf{i})$ is a strict 2-natural transformation.*

Proof. It is well-known that the category $\mathbf{N}(\mathbf{i})$ is an additive \mathbb{C} -linear category, see [Ke, Page 552] and [CM, Section 2]. For each indecomposable object X in $\mathbf{L}(\mathbf{i})$, the object $\Theta(X)$ is not isomorphic to X , moreover, $\text{Hom}_{\mathbf{L}(\mathbf{i})}(X, \Theta(X)) = 0$. From the construction it is now easy to see that the endomorphism algebra of X in $\mathbf{N}(\mathbf{i})$ is the same as in $\mathbf{L}(\mathbf{i})$, in particular, it is local. Therefore X is indecomposable in $\mathbf{N}(\mathbf{i})$ and thus the fact that $\mathbf{N}(\mathbf{i})$ is idempotent split and Krull-Schmidt with finitely many indecomposable objects follows from the corresponding properties in $\mathbf{L}(\mathbf{i})$, by additivity and \mathbb{C} -linearity. This proves claim (i).

Claims (ii) and (iii) follow directly by construction, taking into account the fact that the 2-natural transformation Θ is strict. \square

Proposition 21 gives us the 2-representation \mathbf{N} of \mathcal{Q}_4 . This 2-representation has rank one. Being of rank one, this 2-representation is transitive. As the underlying algebra of $\mathbf{L}(\mathbf{i})$ is $D \oplus D$, from the proof of Proposition 21 we see that the underlying algebra of $\mathbf{N}(\mathbf{i})$ is D . Therefore the restriction of \mathbf{N} to the 2-subcategory \mathcal{A}_4 is simple transitive. This means that \mathbf{N} itself is simple transitive as well.

5.9. Some abstract nonsense on orbit categories. Let G be an abelian group and \mathcal{Q} a category equipped with a (strict) G -action, $g \mapsto F_g : \mathcal{Q} \rightarrow \mathcal{Q}$. Following [CM, Definition 2.3] (see also [As]), we define the *skew category* $\mathcal{Q}[G]$ as follows:

- $\mathcal{Q}[G]$ has the same objects as \mathcal{Q} ;
- $\mathcal{Q}[G](i, j) = \bigoplus_{g \in G} \mathcal{Q}(i, F_g(j))$;
- composition in $\mathcal{Q}[G]$ is given by composition in \mathcal{Q} , after adjustment.

We denote by $T : \mathcal{Q} \rightarrow \mathcal{Q}[G]$ the natural inclusion. Note that, in Subsection 5.8, we have $\mathbf{N}(\mathbf{i}) = \mathbf{L}(\mathbf{i})[G]$, where $G = \{\text{Id}_{\mathbf{L}(\mathbf{i})}, \Theta\}$.

Lemma 22. *For any $g \in G$, there exists a natural isomorphism $\xi(g) : T \cong T \circ F_g$ of functors from \mathcal{Q} to $\mathcal{Q}[G]$ such that $\xi(e)$ is the identity and the following diagram commutes, for all $g, h \in G$:*

$$(13) \quad \begin{array}{ccc} T & \xrightarrow{\xi(gh)} & T \circ F_{gh} \\ \xi(h) \downarrow & & \parallel \\ T \circ F_h & \xrightarrow{\xi(g) \circ \text{id}_{F_h}} & T \circ F_g \circ F_h \end{array}$$

Proof. On any object $i \in \mathcal{Q}$, the value of $\xi(g)$ is given by the identity morphism $\varepsilon_i \in \mathcal{Q}(i, i)$ which is, at the same time, an element in $\mathcal{Q}[G](i, F_g(i))$. It is clear from the construction that $\xi(g)$ defined in this way is a natural isomorphism, that $\xi(e)$ is the identity and that (13) commutes. \square

The following statement is similar to [As, Proposition 2.6].

Proposition 23. *Let \mathcal{Q}' be a category and $K : \mathcal{Q} \rightarrow \mathcal{Q}'$ be a functor. Assume that, for each $g \in G$, there is an isomorphism $\tau(g) : K \rightarrow K \circ F_g$ such that $\tau(e)$ is the identity and the following diagram commutes, for all $g, h \in G$:*

$$(14) \quad \begin{array}{ccc} K & \xrightarrow{\tau(gh)} & K \circ F_{gh} \\ \tau(h) \downarrow & & \parallel \\ K \circ F_h & \xrightarrow{\tau(g) \circ \text{id}_{F_h}} & K \circ F_g \circ F_h \end{array}$$

Then there is a functor $\overline{K} : \mathcal{Q}[G] \rightarrow \mathcal{Q}'$ such that $K = \overline{K} \circ T$.

Proof. On objects and morphisms from $\mathcal{Q}(i, j)$, we define \overline{K} as K . This, in particular, guarantees $K = \overline{K} \circ T$. It remains to define \overline{K} on morphisms from $\mathcal{Q}(i, F_g(j))$, considered as a subset of $\mathcal{Q}[G](i, j)$, where $g \in G$ is different from the identity element.

For every $\alpha \in \mathcal{Q}(i, F_g(j)) \subset \mathcal{Q}[G](i, j)$, we define $\overline{K}(\alpha) := \tau(g)_j^{-1} \circ K(\alpha)$ (this, in particular, agrees with the previous paragraph in the case $g = e$). Now we have to check the functoriality of this construction.

Let $\alpha \in \mathcal{Q}(i, F_g(j)) \subset \mathcal{Q}[G](i, j)$ and $\beta \in \mathcal{Q}(j, F_h(k)) \subset \mathcal{Q}[G](i, k)$, where $g, h \in G$. Then we compute:

$$\begin{aligned}
\overline{K}(\beta) \circ \overline{K}(\alpha) &= \tau(h)_k^{-1} \circ K(\beta) \circ \tau(g)_j^{-1} \circ K(\alpha) && \text{(definition)} \\
&= \tau(h)_k^{-1} \circ \tau(g)_{F_h(k)}^{-1} \circ K(F_g(\beta)) \circ K(\alpha) && \text{(naturality of } \tau(g)) \\
&= \tau(h)_k^{-1} \circ \tau(g)_{F_h(k)}^{-1} \circ K(F_g(\beta) \circ \alpha) && \text{(functoriality of } K) \\
&= \tau(gh)_k^{-1} \circ K(F_g(\beta) \circ \alpha) && \text{(by (14)).}
\end{aligned}$$

This completes the proof. \square

5.10. Proof of Theorem 12. In the rank two case, the combinatorics of the action is described by Lemma 14(b). Consider the additive closure of $\theta_s L_1$ and $\theta_{sts} L_1$. It is stable under the action of \mathcal{Q}_4 . By Lemma 16, both $\theta_s L_1$ and $\theta_{sts} L_1$ are projective modules. A usual argument involving the action of θ_s (cf. Corollary 4) shows that they are indecomposable. Therefore $P_1 \cong \theta_s L_1$ and $P_2 \cong \theta_{sts} L_1$. Now the fact that \mathbf{M} is isomorphic to $\mathbf{C}_{\mathcal{L}_s}$ is proved using arguments similar to the ones in Subsection 4.9.

Let now \mathbf{M} be a simple transitive 2-representation of \mathcal{Q}_4 of rank one. Consider its abelianization $\overline{\mathbf{M}}$ and let Φ be the unique strict 2-natural transformation from \mathbf{P}_1 to $\overline{\mathbf{M}}$ which sends 1_1 to a (unique up to isomorphism) simple object L in $\overline{\mathbf{M}}(\mathbf{i})$. In the same way as we did several times above, Φ restricts to a strict 2-natural transformation from $\mathbf{C}_{\mathcal{L}_s}$ to $\mathbf{M}^{(1)}$ which sends both θ_s and θ_{sts} to indecomposable (but now isomorphic) objects.

Applying abelianization to the above, we get a strict 2-natural transformation from $\mathbf{Q}^{(k)}$ to $\mathbf{M}^{(k+1)}$, for each $k \geq 0$, which is compatible with the canonical inclusions $\mathbf{Q}^{(k)} \hookrightarrow \mathbf{Q}^{(k+1)}$ and $\mathbf{M}^{(k+1)} \hookrightarrow \mathbf{M}^{(k+2)}$. Taking the limit, gives a strict 2-natural transformation Γ from the 2-representation \mathbf{K} from Subsection 5.8 to the limit $\hat{\mathbf{M}}$ of the inductive system

$$\mathbf{M}^{(0)} \longrightarrow \mathbf{M}^{(1)} \longrightarrow \mathbf{M}^{(2)} \longrightarrow \dots$$

Note that \mathbf{K} is equivalent to $\mathbf{C}_{\mathcal{L}_s}$ while $\hat{\mathbf{M}}$ is equivalent to \mathbf{M} . This gives us the solid part of the diagram

$$(15) \quad \begin{array}{ccc} \mathbf{L} & \xrightarrow{\Pi \circ \Upsilon} & \mathbf{K} \\ \Xi \downarrow & & \downarrow \Gamma \\ \mathbf{N} & \xrightarrow{\Delta} & \hat{\mathbf{M}}. \end{array}$$

Let X_1 be an indecomposable object in $\mathbf{L}(\mathbf{i})$. Then $\Theta(X_1)$ is also indecomposable and not isomorphic to X_1 . Together X_1 and $\Theta(X_1)$ generate $\mathbf{L}(\mathbf{i})$ as an additive category. From the construction above we see that the objects $\Gamma \circ \Pi \circ \Upsilon(X_1)$ and $\Gamma \circ \Pi \circ \Upsilon \circ \Theta(X_1)$ are isomorphic in $\hat{\mathbf{M}}(\mathbf{i})$. Fixing an isomorphism between these two object extends to an invertible modification $\nu : \Gamma \circ \Pi \circ \Upsilon \rightarrow \Gamma \circ \Pi \circ \Upsilon \circ \Theta$. As Θ is a strict involution, it follows that the modification ν , together with the identity modification on $\Gamma \circ \Pi \circ \Upsilon$ satisfy all conditions of Proposition 23. In particular, the following diagram establishes (14) in this case:

$$\begin{array}{ccc} \Gamma \circ \Pi \circ \Upsilon & \xrightleftharpoons[\nu^{-1}]{\nu} & \Gamma \circ \Pi \circ \Upsilon \circ \Theta \\ \nu \downarrow & & \downarrow \nu^{-1} \\ \Gamma \circ \Pi \circ \Upsilon \circ \Theta & \xrightleftharpoons[\nu]{\nu^{-1}} & \Gamma \circ \Pi \circ \Upsilon \circ \Theta \circ \Theta. \end{array}$$

From Proposition 23, we thus get the dotted functor Δ in (15) which makes (15) a strictly commutative diagram. As Θ strictly commutes with the action of \mathcal{Q}_4 , from the construction in Proposition 23 we see that Δ is, in fact, a strict 2-natural transformation. By construction, Δ maps a (unique up to isomorphism) indecomposable generator, say X , of $\mathbf{N}(\mathbf{i})$ to a (unique up to isomorphism) indecomposable generator, say Y , of $\hat{\mathbf{M}}(\mathbf{i})$. From our analysis above we know that the endomorphism algebras of both X and Y are isomorphic to D . Restriction of Δ to the action of \mathcal{A}_4 is, therefore, an equivalence as both these restricted 2-representations must be equivalent to the cell 2-representation of \mathcal{A}_4 . This implies that Δ is an equivalence between \mathbf{N} and $\hat{\mathbf{M}}$. Hence \mathbf{N} and \mathbf{M} are equivalent as well. The proof is complete.

REFERENCES

- [As] H. Asashiba. A generalization of Gabriel’s Galois covering functors and derived equivalences. *J. Algebra* **334** (2011), 109–149.
- [BL] A. Bartels, W. Lück. On crossed product rings with twisted involutions, their module categories and L-theory. *Cohomology of groups and algebraic K-theory*, 1–54, Adv. Lect. Math. (ALM), **12**, Int. Press, Somerville, MA, 2010.
- [BFK] J. Bernstein, I. Frenkel, M. Khovanov. A categorification of the Temperley-Lieb algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors. *Selecta Math. (N.S.)* **5** (1999), no. 2, 199–241.
- [ChMa] A. Chan, V. Mazorchuk. Diagrams and discrete extensions for finitary 2-representations. Preprint arXiv:1601.00080.
- [CR] J. Chuang, R. Rouquier. Derived equivalences for symmetric groups and \mathfrak{sl}_2 -categorification. *Ann. of Math. (2)* **167** (2008), no. 1, 245–298.
- [CM] C. Cibils, E. Marcos. Skew category, Galois covering and smash product of a k -category. *Proc. Amer. Math. Soc.* **134** (2006), no. 1, 39–50.
- [E] B. Elias. Soergel Diagrammatics for dihedral Groups. PhD thesis, Columbia University, 2011.
- [EW] B. Elias, G. Williamson. The Hodge theory of Soergel bimodules. *Ann. of Math. (2)* **180** (2014), no. 3, 1089–1136.
- [Ke] B. Keller. On triangulated orbit categories. *Doc. Math.* **10** (2005), 551–581.
- [KL] M. Khovanov, A. Lauda. A diagrammatic approach to categorification of quantum groups. I. *Represent. Theory* **13** (2009), 309–347.
- [KM1] T. Kildetoft, V. Mazorchuk. Parabolic projective functors in type A . Preprint arXiv:1506.07008.
- [KM2] T. Kildetoft, V. Mazorchuk. Special modules over positively based algebras. Preprint arXiv:1601.06975.
- [Le] T. Leinster. Basic bicategories, Preprint arXiv:math/9810017.
- [Mac] S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, 1998.
- [Maz] V. Mazorchuk. Lectures on algebraic categorification. QGM Master Class Series. European Mathematical Society (EMS), Zurich, 2012.
- [MM1] V. Mazorchuk, V. Miemietz. Cell 2-representations of finitary 2-categories. *Compositio Math* **147** (2011), 1519–1545.
- [MM2] V. Mazorchuk, V. Miemietz. Additive versus abelian 2-representations of fiat 2-categories. *Moscow Math. J.* **14** (2014), No. 3, 595–615.
- [MM3] V. Mazorchuk, V. Miemietz. Endomorphisms of cell 2-representations. Preprint arXiv:1207.6236. To appear in IMRN.
- [MM4] V. Mazorchuk, V. Miemietz. Morita theory for finitary 2-categories. *Quantum Topol.* **7** (2016), No. 1, 1–28.
- [MM5] V. Mazorchuk, V. Miemietz. Transitive 2-representations of finitary 2-categories. Preprint arXiv:1404.7589. To appear in *Trans. Amer. Math. Soc.*
- [MM6] V. Mazorchuk, V. Miemietz. Isotypic faithful 2-representations of \mathcal{J} -simple fiat 2-categories. *Math. Z.* **282** (2016), no. 1-2, 411–434.
- [MZ] V. Mazorchuk, X. Zhang. Simple transitive 2-representations for two non-fiat 2-categories of projective functors. Preprint arXiv:1601.00097.
- [Ro] R. Rouquier. 2-Kac-Moody algebras. Preprint arXiv:0812.5023.
- [Se] P. Seidel. Fukaya categories and Picard-Lefschetz theory. *Zurich Lectures in Advanced Mathematics*. European Mathematical Society (EMS), Zürich, 2008.

- [So1] W. Soergel. The combinatorics of Harish-Chandra bimodules. *J. Reine Angew. Math.* **429** (1992), 49–74.
- [So2] W. Soergel. Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen. *J. Inst. Math. Jussieu* **6** (2007), no. 3, 501–525.
- [Xa] Q. Xantcha. Gabriel 2-Quivers for Finitary 2-Categories. *J. London Math. Soc.* **92** (2015), no. 3, 615–632.
- [Zh1] X. Zhang. Duflo involutions for 2-categories associated to tree quivers. Preprint arXiv:1501.03052. To appear in *J. Algebra Appl.*
- [Zh2] X. Zhang. Simple transitive 2-representations and Drinfeld center for some finitary 2-categories. Preprint arXiv:1506.02402.
- [Zi] J. Zimmermann. Simple transitive 2-representations of Soergel bimodules in type B_2 . Preprint arXiv:1509.01441.

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